

LOGARITHM LAWS FOR UNIPOTENT FLOWS, II

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ABSTRACT. We prove analogs of the logarithm laws of Sullivan and Kleinbock-Margulis in the context of unipotent flows. In particular, we prove results for horospherical actions on homogeneous spaces G/Γ . We describe some relations with multi-dimensional diophantine approximation.

1. INTRODUCTION

Two important dynamical systems on non-compact manifolds are the geodesic and horocycle flows on the unit tangent bundle of a finite-volume non-compact hyperbolic surface. Both of these flows are known to be ergodic, and thus, generic orbits are dense. A natural question is to understand the behavior of excursions of trajectories into the cusp(s).

For geodesic flows, the statistical properties of these excursions were first studied in [17] by Sullivan (in the context of finite volume hyperbolic manifolds) and later, in the more general context of the actions of one-parameter diagonalizable subgroups on non-compact finite-volume homogeneous spaces, by Kleinbock-Margulis. In [12], they proved the following result:

Theorem 1.1. ([12], *Theorem 1.7 and Prop 5.1*) *Let G be a connected semisimple Lie group without compact factors, \mathfrak{g} its Lie algebra, $\Gamma \subset G$ an irreducible non-uniform lattice, K a maximal compact subgroup, and $d(\cdot, \cdot)$ a distance function on G/Γ determined by a right-invariant Riemannian metric on G bi-invariant under K . Let μ denote the measure on G/Γ determined by Haar measure on G . Let $\mathfrak{a} \subset \mathfrak{g}$ be a Cartan subalgebra, $0 \neq z \in \mathfrak{a}$, and $a_t = \exp(tz)$. Then there exists a $k = k(G/\Gamma, d) > 0$ such that $\forall y$,*

- $\exists C_1, C_2 > 0$ such that for all $t > 0$,

$$(1.1) \quad C_1 e^{-kt} \leq \mu(x \in G/\Gamma : d(x, y) > t) \leq C_2 e^{-kt}.$$

- For μ -a.e. x ,

$$(1.2) \quad \limsup_{t \rightarrow \infty} \frac{d(a_t x, y)}{\log t} = 1/k.$$

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In this paper, we prove results similar to equation (1.2) for several classes of *unipotent* actions. This paper is a sequel to [4], where we considered the case of unipotent flows on the space of lattices. Subsequently, there has been significant activity in the setting of unipotent logarithm laws, for example the papers [3, 5, 11].

Our results can broadly be divided into two categories:

- (1) *Horospherical actions.* We prove a result (Theorem 2.1) on the excursions of orbits of large subsets of horospherical subgroups. We obtain lower bounds for specific orbits.
- (2) *Almost everywhere results for flows.* This result (Theorem 2.5) applies in the most general situation of one-parameter unipotent flows on symmetric spaces, and uses probabilistic techniques (generalized Borel-Cantelli lemmas) and exponential decay of matrix coefficients.

1.1. Organization: The paper is organized as follows: in §2, we state our main results. In §3, we collect technical results on tori and divergent trajectories required for our proofs. In §4, we use these technical results to prove our main theorem on horospherical actions, as well as related corollaries on hyperbolic surfaces. Finally, in §5, we prove our probabilistic results.

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2. STATEMENT OF RESULTS

2.1. Horospherical actions. Let G be a connected semisimple Lie group without compact factors, and $\Gamma \subset G$ be an irreducible non-uniform lattice. Let μ denote the probability measure on G/Γ arising from Haar measure on G . Let A be a maximal connected \mathbb{Q} -diagonalizable subgroup, and $\{a_t\}_{t \in \mathbb{R}}$ be a one-parameter subgroup of A , and let

$$(2.1) \quad H := \{h \in G : a_{-t}ha_t \rightarrow_{t \rightarrow +\infty} 1\}$$

be the *expanding* horospherical subgroup associated to $\{a_t\}$.

Given $x_0 \in G/\Gamma$, Dani ([7], Theorem 1.6) proved that if $\{a_{-t}x_0\}_{t \geq 0}$ is non-divergent (i.e., there is a compact set $C \in G/\Gamma$ and a sequence of times $t_n \rightarrow +\infty$ such that $a_{-t_n}x_0 \in C$), that Hx_0 is dense in G/Γ . Our aim is to give a more quantitative version of this result, with regards to visits to neighborhoods of ∞ . Let $B \subset H$ be a non-empty, bounded, open subset. Set

$$(2.2) \quad B_t := a_{\log t} B a_{-\log t}.$$

This forms an expanding family of subsets of H .

Let d_X denote a right-invariant metric on G arising from the Riemannian metric induced by the Killing form. If $G' \subset G$ is a subgroup of G , we let $d_{G'}$ denote the induced distance function on G' . We let $d_{G/\Gamma}$ denote the induced distance function on G/Γ . We will drop the subscripts when it is clear on

which space we are measuring distances. We will study the behavior of the excursions of $B_t x_0$ away from compact sets by investigating the asymptotic behavior of the quantities

$$(2.3) \quad \beta_t(x_0) := \sup_{b \in B_t} d_{G/\Gamma}(bx_0, x_0).$$

Since Hx_0 is dense for all x_0 such that $\{a_{-t}x_0\}_{t \geq 0}$ is non-divergent, we have

$$\limsup_{t \rightarrow \infty} \beta_t(x_0) = \infty$$

for such x_0 . Our main result is about the rate of these excursions.

To formulate our results, we need a little more notation. If the \mathbb{R} -rank of G is at least 2, we can assume, by the Arithmeticity Theorem ([14], Chapter IX), that $G = \mathbb{G}(\mathbb{R})^\circ$ and $\Gamma = \mathbb{G}(\mathbb{Z})$, where \mathbb{G} is a semisimple algebraic \mathbb{Q} -group. Let \mathbb{S} be a maximal \mathbb{Q} -split torus. Without loss of generality, we can assume $A = \mathbb{S}(\mathbb{R})^\circ$. Let $\|\cdot\|$ denote the norm on A induced by the Killing form. We can write $a_t = \exp t\mathbf{z}$, with $\mathbf{z} \in \mathfrak{a}$. If the \mathbb{R} -rank of G is equal to 1, then there is (up to scaling and conjugation) a unique 1-parameter subgroup, which we again can write as $a_t = \exp t\mathbf{z}$ for $z \in \mathfrak{a}$, where \mathfrak{a} is the Lie algebra to $\{a_t\}$.

Given $x_0 \in G/\Gamma$, let

$$\omega^- := \omega^-(x_0, a_t, d, \Gamma) := \limsup_{t \rightarrow +\infty} \frac{d_{G/\Gamma}(a_{-t}x_0, x_0)}{t}.$$

Theorem 2.1. *Fix notation as above. Let $a_t = \exp(t\mathbf{z})$, $\mathbf{z} \in \mathfrak{a}$. Let $\nu = \|\mathbf{z}\|$.*

$$(2.4) \quad \limsup_{t \rightarrow \infty} \frac{\beta_t(x_0)}{\log t} \leq \nu + \omega^-$$

If $\{a_{-t}x_0\}_{t \geq 0}$ is non-divergent, then

$$(2.5) \quad \limsup_{t \rightarrow \infty} \frac{\beta_t(x_0)}{\log t} \geq \nu$$

We will prove this theorem in §4. Combining this result with Theorem 1.1, which implies that $\omega^-(x_0) = 0$ for μ -almost every $x_0 \in G$, we have the following corollary:

Corollary 2.2. *Fix notation as above.*

$$\limsup_{t \rightarrow \infty} \frac{\beta_t(x_0)}{\log t} = \nu$$

for μ -almost every x_0 .

Remarks:

- It is initially somewhat surprising that the typical horospherical excursion should have the same order as the typical excursion for a_t . What we will show in the proof is that the behavior of the horosphere is governed by divergent, non-typical, a_t -trajectories.
- (2.4) follows relatively easily from the triangle inequality, whereas (2.5) requires a more detailed analysis of divergent $\{a_t\}$ -trajectories.
- We expect that our results will hold for general *norm-like pseudo-metrics*, as defined in [1].

2.2. Hyperbolic surfaces. Specializing to $G = SL(2, \mathbb{R})$ with $H = \{h_s\}_{s \in \mathbb{R}}$, where

$$(2.6) \quad h_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

we have that

$$(2.7) \quad a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}.$$

We take $B = \{h_s\}_{s \in (0,1)}$, and so $B_t = \{h_s\}_{s \in (0,t)}$ ($a_t h_s a_{-t} = h_{set}$), and obtain a sharp result for the horocycle flow on the unit tangent bundle of a general non-compact finite volume hyperbolic surface. Let $\Gamma \subset SL(2, \mathbb{R})$ be a non-uniform lattice. Let d denote distance on the hyperbolic surface $S = \mathbb{H}^2/\Gamma$ (\mathbb{H}^2 denotes the upper-half plane with constant curvature -1), and $p : M \rightarrow S$ be the natural projection from $M = SL(2, \mathbb{R})/\Gamma$.

Corollary 2.3. *Let $H = \{h_s\}_{s \in \mathbb{R}}$. Fix $y \in S$. Then for all $x \in S$, almost all $\tilde{x} \in p^{-1}(x)$,*

$$(2.8) \quad \limsup_{s \rightarrow \infty} \frac{d(p(h_s \tilde{x}), y)}{\log s} = 1.$$

Moreover, for all $\tilde{x} \in M$ such that $H\tilde{x}$ is not closed,

$$(2.9) \quad \limsup_{s \rightarrow \infty} \frac{d(p(h_s \tilde{x}), y)}{\log s} \geq 1.$$

The following proposition shows that while (2.8) holds for almost every point, the inequality in (2.9) is strict for a (topologically) large set of points:

Proposition 2.4. *Let $\Gamma \subset SL(2, \mathbb{R})$ be a non-uniform lattice, $H = \{h_s\}_{s \in \mathbb{R}}$ as in equation 2.6. Let $y \in \mathbb{H}^2/\Gamma$. Let*

$$B = \{x \in SL(2, \mathbb{R})/\Gamma : \limsup_{t \rightarrow \infty} \frac{d(p(h_t x), y)}{\log t} = 2\}.$$

B contains a dense set of second Baire category.

Remarks:

- Note that in the metric on \mathbb{H}^2 , $d(p(h_s), i) = 2 \log s$ (where by abuse of notation, $p : SL(2, \mathbb{R}) \rightarrow \mathbb{H}^2 = SO(2) \backslash SL(2, \mathbb{R})$ is the projection $p(g) = SO(2)g$), so 2 is the maximum value this lim sup can attain. In fact, for any sequence $r_n \rightarrow \infty$ in $SL(2, \mathbb{R})$, $d(p(r_n), i) \approx 2 \log |r_n|$ (\approx means the ratio goes to 1), where $|g|$ is the supremum of the matrix entries of g .
- By (2.4), the set B must consist of trajectories which diverge at rate 1 under a_{-t} , that is, they must satisfy

$$\limsup_{t \rightarrow \infty} \frac{d(p(a_{-t}\tilde{x}), y)}{t} = 1.$$

- In [4], we consider the special case of $\Gamma = SL(2, \mathbb{Z})$, and obtain several connections to Diophantine approximation. Further results can be found in [3], in which precise conditions for this lim sup to take on certain values for general $SL(2, \mathbb{R})/\Gamma$ are given, and in [11] where results are obtained for quotients of products of $SL(2, \mathbb{R})$ and $SL(2, \mathbb{C})$.

2.3. Upper and lower bounds. We now return to the case of general semisimple Lie groups G and non-uniform lattices Γ . Now we study the action of *one-parameter* unipotent subgroups on G/Γ . We have

Theorem 2.5. *Fix notation as in Theorem 1.1. Let $\{u_t\}_{t \in \mathbb{R}} \subset G$ denote a one-parameter unipotent subgroup. Then there is a $0 < \alpha \leq 1$ such that for $\forall y$, μ -a.e. x ,*

$$\limsup_{t \rightarrow \infty} \frac{d(u_t x, y)}{\log t} = \alpha/k$$

We prove this theorem in §5.

Remark: Note that Theorem 1.1 says that if we replace our unipotent subgroup $\{u_t\}$ with a diagonalizable subgroup $\{a_t\}$, we can always take $\alpha = 1$. Like Theorem 1.1, Theorem 2.5 is proved using information on decay of matrix coefficients of the regular representation of G on G/Γ , and an appropriately adapted version of the Borel-Cantelli lemma. However, the slower decay of matrix coefficients for unipotent flows as compared to diagonalizable flows does not allow us to conclude that $\alpha = 1$. It would be very interesting to find examples of unipotent subgroups where $\alpha \neq 1$, though we suspect that such subgroups do not exist.

3. DIVERGENT TRAJECTORIES

Recall the notation of §2.1: G is a connected semisimple Lie group without compact factors, Γ an irreducible non-uniform lattice, and d the Riemannian metric arising from the Killing form on G . We also use d to denote the metric

on G/Γ . A is a maximal \mathbb{Q} -diagonalizable subgroup of G , and $\{a_t\}$ a one-parameter subgroup of A . Write $a_t = \exp t\mathbf{z}$, $\mathbf{y} \in \mathfrak{a}$, and let $\nu = \|y\|$. We prove the following result concerning $\{a_t\}$ trajectories in G/Γ .

Proposition 3.1. *Fix $x_0 \in G/\Gamma$. For all $x \in G/\Gamma$*

$$(3.1) \quad \limsup_{t \rightarrow \infty} \frac{d(a_t x, x_0)}{t} \leq \nu.$$

Moreover, for all $x = g\Gamma$ with $g \in \mathbb{G}(\mathbb{Q})$,

$$(3.2) \quad \lim_{t \rightarrow \infty} \frac{d(a_t x, x_0)}{t} = \nu.$$

3.1. Reduction Theory. We recall some results from reduction theory. Assume the \mathbb{R} -rank of G is greater than 1. We can assume, as in §2.1, $G = \mathbb{G}(\mathbb{R})^\circ$ and $\Gamma = \mathbb{G}(\mathbb{Z})$, where \mathbb{G} is a semisimple algebraic \mathbb{Q} -group. Let \mathbb{S} be a maximal \mathbb{Q} -split torus in \mathbb{G} , and set $A = \mathbb{S}(\mathbb{R})^\circ$.

Let Φ be a system of \mathbb{Q} -roots associated to \mathfrak{A} and let Φ^+ and Φ^s be the sets of positive and simple roots respectively. We define the positive *Weyl chamber*

$$\mathfrak{a}^+ = \{z \in \mathfrak{d} : \alpha(z) \geq 0 \text{ for all } \alpha \in \Phi^s\}.$$

Using the exponential map, we identify it with $A^+ = \exp(\mathfrak{a}^+) \subset A$. Conjugating if necessary, we can assume that $\mathbf{z} \in \mathfrak{a}^+$, that is, $a_t \in A^+$ for $t > 0$. We have the Iwasawa decomposition $G = KAMU$ (here, K is a maximal compact subgroup, U is unipotent, and M is reductive, with A centralizing M and normalizing U). Let $Q \subset MU$ be relatively compact, and for $\tau > 0$, define

$$\mathfrak{a}_\tau = \{z \in \mathfrak{a} : \alpha(z) \geq \tau \text{ for all } \alpha \in \Phi^s\}$$

We can define a *generalized Siegel set*

$$S_{Q,\tau} := K \exp(\mathfrak{a}_\tau) Q.$$

For appropriate choices of Q and τ , a finite union of translates of $S_{Q,\tau}$ form a weak fundamental domain for the Γ -action on G . Precisely, we have

Theorem 3.2 ([13], Proposition 2). *Fix notation as above. There is are Q, τ and $\{q_1, \dots, q_m\} \in \mathbb{G}(\mathbb{Q})$ so that $\Omega := \bigcup_{i=1}^m S_{Q,\tau} q_i$ satisfies*

- (1) $G = \Omega\Gamma$
- (2) *For all $q \in \mathbb{G}(\mathbb{Q})$, $\{\gamma \in \Gamma : \Omega q \cap \Omega\gamma \neq \emptyset\}$ is finite.*

The finite set q_1, \dots, q_m form a set of representatives for the double coset space $\mathbb{P}(\mathbb{Q}) \backslash \mathbb{G}(\mathbb{Q}) / \Gamma$, where $\mathbb{P}(\mathbb{Q})$ is a minimal \mathbb{Q} -parabolic subgroup. Note that $\mathbb{P}(\mathbb{R})^\circ = AMU$. Fix x_0 to be the identity coset in G/Γ . Let $x = g\Gamma$. Note that, letting e denote the identity in G , we have, for any $h \in G$,

$$d(hx, x_0) = \inf_{\gamma \in \Gamma} d(hg, \gamma)$$

Using Theorem 3.2, Leuzinger [13, Theorem 1] proved that there is a $b \in \overline{A^+}$ (here, $\overline{A^+}$ denotes the closure of the Weyl chamber A^+) such for any $\mathbf{y} \in \mathfrak{a}^+$, with $\|\mathbf{y}\| = \mathbf{1}$, $a_t := \exp(t\mathbf{y})$, any $p \in MU$, any $q_i, 1 \leq i \leq m$, and $\gamma \in \Gamma$, we have

$$(3.3) \quad d(a_t b p q_i, p b q_i \gamma) \geq t$$

3.2. Proof of Proposition 3.1. First note that the upper bound (3.1) follows from the definition of distance on the quotient and the fact that $\|\mathbf{z}\| = \nu$. To show the limit result (3.2), we first consider the case when \mathbb{R} -rank is at least 2, so we can apply Theorem 3.2 and equation (3.3). Since we can write each element $g \in \mathbb{G}(\mathbb{Q})$ as $g = p q_i \gamma_0$ for $p \in \mathbb{P}(\mathbb{Q})$ and $\gamma \in \Gamma$, and denoting the bounded error given by the element b by C , we have, for all $\gamma \in \Gamma$,

$$(3.4) \quad d(a_t g, \gamma) \geq \nu t - C$$

(3.2) follows immediately.

3.2.1. \mathbb{R} -rank 1. Finally, suppose the \mathbb{R} -rank of G is 1. Applying standard reduction theory [8] and the density of orbits of parabolic subgroups ([16], Lemma 8.5) there is a dense set of points diverging under a_t at rate $\nu = \|z\|$. See also [7, 18] for more details on divergent trajectories. \square

4. HOROSPHERICAL ACTIONS

We fix notation as in §2.1 and §3. The proof of Theorem 2.1 splits naturally into an upper and lower bound:

4.1. Lower bound.

Lemma 4.1. *For all $x \in G/\Gamma$ with $\{a_{-t}x\}_{t>0}$ non-divergent,*

$$(4.1) \quad \limsup_{t \rightarrow \infty} \frac{\beta_t(x)}{\log t} \geq \nu.$$

Proof: The idea is as follows: given the piece of orbit $B_{e^T}x$, we want to show that it has moved depth T into the cusp. We can write $B_{e^T}x = a_t B a_{-t}x$. If $a_{-t}x$ is non-divergent, we can take some T so that $a_{-t}x$ is in a compact set. Using the fact the forward divergent $\{a_t\}$ trajectories are dense, we can find a divergent trajectory (moving at rate ν) in a ‘thickening’ of the orbit $B a_{-t}x$ in the directions transverse to H . Since a_t does not expand the directions transverse to H , the divergent trajectory (which will be approximately depth νT into the cusp after applying a_t) will be near $B_{e^T}x$, so there is some $h \in B_{e^T}$ with hx almost depth T into the cusp, as desired. To make this argument precise, we need to use the following

Lemma 4.2. *Let $C \subset G/\Gamma$ be compact with non-empty interior, and $\epsilon, \phi > 0$. Then there is a $T_{C,\epsilon,\phi}$ such that*

$$\{x : d(a_t x, x_0) > (\nu - \phi)t - T_{C,\epsilon,\phi} \text{ for all } t > 0\}$$

is ϵ -dense in C .

Proof: Note that by Prop 3.1,

$$\{x \in G/\Gamma : \exists T(x) \text{ such that } d(a_t x, x) > (\nu - \phi)t - T(x) \text{ for all } t > 0\}$$

is dense in G/Γ .

Now let $\epsilon > 0$, $C \subset G/\Gamma$ compact. Let $\{B(x, \epsilon)\}_{x \in C}$ be the cover of C by open ϵ -metric balls. Since C is compact, we can take a finite subcover $\{D_1, D_2, \dots, D_n\}$, where each $D_i = B(x_i, \epsilon)$. For $1 \leq i \leq n$, there is a $x_i \in D_i$, $T(x_i) > 0$, such that

$$d(a_t x_i, y) > (\nu - \phi)t - T(x_i).$$

Let $T_{C,\epsilon,\phi} = \max_{1 \leq i \leq n} T(x_i)$. Now, for all $x \in C$, there is an x_i such that $d(x_i, x) < \epsilon$, and for all $1 \leq i \leq n$, $d(a_t x_i, y) > (\nu - \phi)t - T_{C,\epsilon,\phi}$, so we have our result. \square

Let H^{-0} be the subgroup associated to the neutral/stable directions for a_t ($t > 0$). Let $x \in G/\Gamma$ be such that $a_{-t}x$ is non-divergent. Thus, there is a compact $C'' \subset G/\Gamma$ be compact with a non-empty interior and $t_n \rightarrow \infty$ so that $a_{-t_n}x \in C''$ for all n .

We fix one more piece of notation: letting G' be a subgroup of G , $g_0 \in G'$, $r > 0$, we let

$$B_{G'}(g_0, r) := \{g \in G' : d_{G'}(g_0, g) < r\}.$$

Let ϵ_1 be such that for all $\epsilon < \epsilon_1$, there are ϵ^+, ϵ^- ,

$$B_G(\epsilon) = B_{H^{-0}}(\epsilon^-)B_H(\epsilon^+).$$

Let $C' = \overline{BC''}$. Let $b_0 \in B$, $\epsilon_0 > 0$ such that $B_H(b_0, \epsilon_0) = B_H(\epsilon_0)b_0 \subset B$. There is an $0 < \epsilon < \epsilon_1$, and an ϵ' so that (perhaps shrinking ϵ_0) we can write

$$B_G(\epsilon) = B_{H^{-0}}(\epsilon')B_H(\epsilon_0).$$

Let $C = \overline{B_G(\epsilon)C'}$. Now $b_0 x_n \in C'$, so $B_G(\epsilon)b_0 x_n \in C$. Shrinking ϵ if necessary, we have

$$B_G(\epsilon)b_0 x_n = B(b_0 x_n, \epsilon).$$

Fix $\phi > 0$, and let $T = T_{C,\epsilon,\phi}$. There is an $x'_n \in B(b_0 x_n, \epsilon)$ so that

$$d(a_{t_n} x'_n, x'_n) > (\nu - \phi)t_n - T.$$

We can write $x'_n = h^- b_0 x_n$ for $h^- \in B_{H^{-0}}(\epsilon')$. Now we have

$$a_{t_n} x'_n = h_n^- b_n x,$$

where

$$h_n^- = a_{t_n} h^- a_{-t_n} \in B_{H^{-0}}(\epsilon')$$

and

$$b_n = a_{t_n} b_0 a_{-t_n} \in B_{e^{t_n}}.$$

Thus, we have

$$d(b_n x, y) \geq d(a_{t_n} x'_n, x'_n) - \epsilon \geq (\nu - \phi)t_n - T - \epsilon$$

So

$$\lim_{n \rightarrow \infty} \frac{d(b_n x, x)}{t_n} \geq (\nu - \phi)$$

as $n \rightarrow \infty$, (note that since x'_n varies in a compact set, it does not matter in the limit whether we measure distance from x or x'_n). Thus,

$$\lim_{n \rightarrow \infty} \frac{\beta_{e^{t_n}}(x)}{t_n} \geq \nu - \phi$$

which, since $\phi > 0$ was arbitrary yields our result. \square

4.2. Upper bound.

Lemma 4.3. *For all $x \in G/\Gamma$,*

$$(4.2) \quad \limsup_{t \rightarrow \infty} \frac{\sup_{h \in B_t} d(hx, y)}{\log t} \leq \nu + \omega(x).$$

Proof: Let $\epsilon > 0$. By the definition of ω , and the boundedness of B for all t sufficiently large, for all $b \in B$,

$$d(ba_{-\log t} x, x) < (\omega + \epsilon) \log t.$$

By definition

$$d(a_{\log t} ba_{-\log t} x, ba_{-\log t} x) \leq \nu \log t.$$

Combining these two inequalities, and using the triangle inequality, we have, for all $b \in B$ and t sufficiently large,

$$d(a_{\log t} ba_{-\log t} x, x) < (\omega + \nu + \epsilon) \log t.$$

Since ϵ was arbitrary, we have our result. \square

Proof of Theorem 2.1: Combine Lemmas 4.3 and 4.1 \square

4.3. Hyperbolic Geometry. In this subsection we prove Corollary 2.8 and Proposition 2.4.

Proof of Corollary 2.8: Apply Theorem 2.1 to $G = SL(2, \mathbb{R})$, with H, B and $\{a_t\}$ as in §2.2. Note that the Riemannian metric on $\mathbb{H}^2 = K \backslash G$ is coarsely isometric to the normlike metric induced by the norm it induces on A . \square

Proof of Proposition 2.4:

We need the following lemma, which exploits properties of divergent geodesic trajectories:

Lemma 4.4. *Let $E \subset SL(2, \mathbb{R})/\Gamma$ be open, and $y \in \mathbb{H}^2/\Gamma$. There is a $C = C(E)$ such that for all $T > 0$ there is a $z \in E$, $t > T$ such that*

$$(4.3) \quad d(p(h_t z), y) > 2 \log t - C$$

Proof: By the density of divergent *geodesic* trajectories there is a $c = c(A) > 0$ and a $z \in E$ such that $d(p(g_s z), y) > s - c$ for all $s > 0$ (for the rest of this section, we will use the notation $a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$).

Fix lifts of y and z to a fundamental domain for Γ in \mathbb{H}^2 , call them y_0 and z_0 (y_0 will be a point, z_0 will be a point and a unit tangent vector). There will be a horocycle connecting $p(z_0)$ and $p(g_s z_0)$, and as $s \rightarrow \infty$, the inward pointing tangent vector to this will approach the vector z_0 .

More precisely, suppose without loss of generality z_0 is i with the upward pointing tangent vector, i.e., $z_0 = e \in SL(2, \mathbb{R})$. Then $p(g_s z_0) = p(g_s) = SO(2)g_s$, and if $v_s = r_{\theta_s} \in SO(2)$ is the unit tangent vector (based at $i = p(z_0)$) determining the horocycle connecting $e^s i = p(g_s)$ and i , we have that v_s approaches the upward pointing tangent vector as $t \rightarrow \infty$, or equivalently $\theta_s \rightarrow 0$.

In addition, if $t = t_s$ is the time it takes for the horocycle to reach $e^s i$, we have $SO(2)g_s = SO(2)h_t r_{\theta_s}$, i.e., there is a θ'_s such that $h_t = r_{\theta'_s} g_s r_{\theta_s}$ (this is simply the Cartan (or *KAK*) decomposition). It is an easy calculation that $t_s \approx e^{s/2}$, or equivalently, $s \approx 2 \log t_s$. Thus, for $s \gg 0$, $r_{\theta_s} z \in A$, and

$$d(p(h_{t_s} r_{\theta_s} z), y) = d(p(g_s z), y) > s - c > 2 \log t_s - C,$$

for some possibly larger C . \square

To complete the proof of the proposition, define $f_T : SL(2, \mathbb{R})/\Gamma \rightarrow [0, 2]$ by

$$f_T(x) = \sup_{2 \leq t \leq T} \frac{d(p(h_t x), y)}{\log t}.$$

$f_T(x)$ is increasing in T , and bounded, so we can define $f_\infty(x) = \lim_{T \rightarrow \infty} f_T(x)$. The f_T 's are continuous for $T < \infty$, but f_∞ is not. We have

$$B = \{x : f_\infty(x) = 2\} = \bigcap_{k=1}^{\infty} \bigcup_{n=0}^{\infty} \left\{x : f_n(x) > 2 - \frac{1}{k}\right\}.$$

Now, for each k , $\bigcup_{n=0}^{\infty} \{x : f_n(x) > 2 - \frac{1}{k}\}$ is dense by Lemma 4.4, and open by the continuity of f_n . Thus B is a countable intersection of open dense sets, as desired. \square

5. BOREL-CANTELLI LEMMAS

In this section we prove Theorem 2.5, using a generalization of the Borel-Cantelli lemma. The classical Borel-Cantelli lemma is as follows:

Lemma 5.1. (*Borel-Cantelli*) Let $\{X_n\}_{n=0}^{\infty}$ be a sequence of 0 – 1 random variables, with $P(X_n = 1) =: p_n$. Then, if $\sum_{n=0}^{\infty} p_n < \infty$,

$$P\left(\sum_{n=0}^{\infty} X_n = \infty\right) = 0.$$

If the X_n 's are pairwise independent, we have that

$$P\left(\sum_{n=0}^{\infty} X_n = \infty\right) = 1.$$

if $\sum_{n=0}^{\infty} p_n = \infty$.

The first statement (non-independent) statement is the ‘easy half’ of this Lemma, and can be derived by simply doing an expectation calculation.

The first example of a logarithm law can be derived from the lemma as follows. Fix $\lambda > 0$. Let $\{Y_n\}_{n=0}^{\infty}$'s be independent identically distributed (i.i.d.) exponential random variables with parameter λ . That is, for any $t > 0$,

$$P(Y_n > t) = e^{-\lambda t}.$$

Let $\{r_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers. Applying Lemma 5.1 to the sequence of random variables

$$X_n := \begin{cases} 1 & Y_n > r_n \\ 0 & \text{otherwise,} \end{cases}$$

implies $Y_n > r_n$ infinitely often if and only if $\sum_{n=0}^{\infty} e^{-\lambda r_n} = \infty$. As a corollary, one obtains that almost surely

$$\limsup_{n \rightarrow \infty} \frac{Y_n}{\log n} = 1/\lambda.$$

To prove Theorem 2.5, we use the following (relatively standard) generalization of Lemma 5.1 to weakly dependent sequences.

Proposition 5.2. *Let (S, Ω, P) be a probability space (i.e., Ω is a σ -algebra of subsets of S , and $P : \Omega \rightarrow [0, 1]$ is a probability measure). Let $X_n : S \rightarrow \{0, 1\}$ be a sequence of 0–1 random variables on S , with $P(X_n = 1) =: p_n$. Also define $p_{i,j} := P(X_i X_j = 1)$. Suppose*

- (1) $\sum_{n=1}^{\infty} p_n = \infty$.
- (2) *There is a function $\psi(m)$ such that for all $m > 0$,*

$$\sup_n |p_{n,n+m} - p_n p_{n+m}| \leq \psi(m).$$

- (3)

$$\lim_{n \rightarrow \infty} \frac{\sum_{m=1}^n \psi(m)(n-m)}{(\sum_{i=1}^n p_i)^2} = 0.$$

Then

$$P\left(\sum_{n=0}^{\infty} X_n = \infty\right) = 1.$$

Proof: Given measurable $X : S \rightarrow \mathbb{R}$, we write $E(X) := \int_S X dP$ for the expectation, and $V(X) = E(X^2) - E(X)^2$ for the variance.

Let $J_n = \sum_{i=1}^n X_i$, and

$$Y_n = \frac{J_n}{\sum_{i=1}^n p_i} = \frac{J_n}{E(J_n)}.$$

We will show that for any $\epsilon > 0$, $P(|Y_n - 1| > \epsilon) \rightarrow 0$, which will imply that there is a sequence n_k such that $Y_{n_k} \rightarrow 1$ with probability 1, and thus, that $J_{n_k} \rightarrow \infty$.

Since $E(Y_n) = 1$, it suffices to show that $V(Y_n) \rightarrow 0$, that is, that $Y_n \rightarrow 1$ in L^2 . Now,

$$V(Y_n) = \frac{V(J_n)}{E(J_n)^2}.$$

We have

$$V(J_n) = V\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n V(X_i) + \sum_{1 \leq i \neq j \leq n} \text{Cov}(X_i, X_j),$$

where $\text{Cov}(X_i, X_j) := |p_{i,j} - p_i p_j|$ is the *covariance* of X_i and X_j .

Now, $\text{Cov}(X_i, X_j) \leq \psi(|j - i|)$ by property (2), so we get that

$$\begin{aligned} V(J_n) &\leq \sum_{i=1}^n p_i + 2 \sum_{i=1}^n \sum_{j=i+1}^n \psi(j-i) = \sum_{i=1}^n p_i + 2 \sum_{i=1}^n \sum_{m=1}^{n-i} \psi(m) \\ (5.1) \qquad \qquad \qquad &= \sum_{i=1}^n p_i + 2 \sum_{m=1}^n \psi(m)(n-m). \end{aligned}$$

Dividing by $(\sum_{i=1}^n p_i)^2$, we get that the two right hand terms go to zero (by properties (1) and (3) respectively), and thus, we have our result. \square

We would like to apply this result to the context of group actions on homogeneous spaces. Fixing notation as in §2.1, and given $y \in G/\Gamma$, we define a sequence of functions $Y_n : G/\Gamma \rightarrow \mathbb{R}^+$ by $Y_n(x) = d(u_n x, y)$. Given a sequence of numbers $\{r_n\}_{n \in \mathbb{N}}$, we set

$$X_n(x) := \begin{cases} 1 & Y_n(x) > r_n \\ 0 & \text{otherwise.} \end{cases}$$

In order to apply Proposition 5.2 to our context, we must estimate two quantities:

- (1) $\mu(x : d(x, y) > t)$
- (2) The covariances for the random variables X_n .

The first estimate follows from equation (1.1), which yields (since u_n is measure preserving):

$$C_1 e^{-kr_n} \leq p_n = \mu(x : X_n(x) = 1) \leq C_2 e^{-kr_n}.$$

In order to estimate the covariances, we must control the matrix coefficients of the sequence $\{u_k\}$ under the regular representation of G on G/Γ . To

do this, we turn once again to [12]. The following result is essentially a combination of Proposition 4.2 and Corollary 3.5 from that paper:

Proposition 5.3. [12] *There are constants $C > 0$, $0 < \beta < 1$ such that for all $n, m \in \mathbb{N}$*

$$|p_{n,n+m} - p_n p_{n+m}| \leq C p_n p_{n+m} m^{-\beta},$$

where $p_{i,j} = \mu(x : X_i(x)X_j(x) = 1)$.

Remark: If we were able to obtain $\beta \geq 2$, we would in fact be able to prove $\alpha = 1$ in the statement of Theorem 2.5 following Prop 4.1 in [12]. However, for reasons beyond the scope of this paper, $\beta < 2$.

We will not prove Proposition 5.3 in this paper, instead referring the interested reader to the appropriate sections of [12].

Proof of Theorem 2.5: Let $r_n > \frac{1}{k} \log n$. Then p_n is summable, so for almost all x , $X_n = 1$ only finitely often, yielding our upper bound. For our lower bound we apply Proposition 5.2 to our sequence X_n , with $\psi(m) = m^{-\beta}$. It is a simple calculation that for any $\gamma < \beta/2$, setting $r_n = \frac{\gamma}{k} \log n$ will yield:

$$\lim_{n \rightarrow \infty} \frac{\sum_{m=1}^n \psi(m)(n-m)}{(\sum_{i=1}^n p_i)^2} = 0.$$

Using Proposition 5.2, we have, for μ -a.e. x ,

$$\beta/2k \leq \frac{\limsup_{t \rightarrow \infty} d(u_t x, y)}{\log t} \leq 1/k.$$

Finally, note that

$$\limsup_{t \rightarrow \infty} \frac{d(u_t x, y)}{\log t}$$

is a measurable u_t -invariant function on G/Γ . Thus, if the u_t -action is ergodic, it must be constant almost everywhere. If u_t is not ergodic, it must act trivially in some factor of G by the Moore ergodicity theorem [15], and thus we can reduce to the ergodic case. \square

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